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Anisotropic inharmonic Higgs oscillator and related (MICZ-)Kepler-like systems

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Abstract

We propose the integrable (pseudo)spherical generalization of the four-dimensional anisotropic oscillator with additional nonlinear potential. Performing its Kustaanheimo–Stiefel transformation we then obtain the pseudospherical generalization of the MICZ-Kepler system with linear and $\cos\theta$ potential terms. We also present the generalization of the parabolic coordinates, in which this system admits the separation of variables. Finally, we get the spherical analog of the presented MICZ-Kepler-like system.

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1. Introduction

The oscillator and Kepler systems are the best-known examples of mechanical systems with hidden symmetries [1]. Due to the existence of hidden symmetry these systems admit separation of variables into few coordinate systems. Despite their qualitative difference, they can be related to each other in some cases. Namely, the $(p + 1)$ -dimensional Kepler system can be obtained by the appropriate reduction procedures from the $2p$ -dimensional oscillator for $p = 1, 2, 4$ (for the review see, e.g., [2]). These procedures, which are known as Levi-Civita (or Bohlin) [3], Kustaanheimo–Stiefel [4] and Hurwitz [5] transformations imply the reduction of the oscillator by the action of Z_2 , $U(1)$, $SU(2)$ groups, respectively, and yield, in general case, the Kepler-like systems with monopoles [6–8]. The second system (with $U(1)$ (Dirac) monopole) is best known and most important among them. It was invented independently by Zwanziger and by McIntosh and Cisneros [9], and presently is referred to as the MICZ-Kepler system.

There are a few deformations of oscillator and Kepler systems, which preserve part of hidden symmetries, e.g. the anisotropic oscillator, the Kepler system with additional linear potential, the two-center Kepler system [1], as well as their ‘MICZ-extensions’ [10]. The Kepler system with a linear potential is of special importance due to its relevance to the Stark

effect. One can observe that the four-dimensional oscillator and with an additional anisotropic term

$$U_A = \frac{\Delta\omega^2}{2} \sum_{\alpha=1,2} (x_\alpha^2 - x_{\alpha+2}^2) \tag{1}$$

results in the (MICZ-)Kepler system with the potential

$$V_{\cos} = \frac{\Delta\omega^2}{4} \cos \theta = \frac{\Delta\omega^2}{4} \frac{x_3}{|\mathbf{x}|}, \tag{2}$$

which is the textbook example of the deformed Kepler system admitting the separation of variables into parabolic coordinates. While the (three-dimensional) Kepler system with an additional linear potential (which is also separable in parabolic coordinates) is originated in the (four-dimensional) oscillator system with the fourth-order anisotropic potential term

$$U_{\text{nlm}} = -2\epsilon_{\text{cl}} \sum_{\alpha=1,2} (x_\alpha^4 - x_{\alpha+2}^4). \tag{3}$$

The corresponding potentials in other dimensions look similarly.

Oscillator and Kepler systems admit the generalizations on a d -dimensional sphere and a two-sheet hyperboloid (pseudosphere). They are defined, respectively, by the following potentials [12, 13],

$$U_{\text{osc}} = \frac{\omega^2 R_0^2}{2} \frac{\mathbf{x}^2}{x_0^2}, \quad V_{\text{Kepler}} = -\frac{\gamma}{R_0} \frac{x_0}{|\mathbf{x}|}, \tag{4}$$

where \mathbf{x}, x_0 are the Cartesian coordinates of the ambient (pseudo)Euclidean space $\mathbb{R}^{d+1}(\mathbb{R}^{d,1})$: $\epsilon \mathbf{x}^2 + x_0^2 = R_0^2, \epsilon = \pm 1$. The $\epsilon = +1$ corresponds to the sphere and $\epsilon = -1$ to the pseudosphere. These systems also possess nonlinear hidden symmetries providing them with the properties similar to those of conventional oscillator and Kepler systems. Various aspects of these systems were investigated in [14]. Let us note and also mention [15], where the integrability of the spherical two-center Kepler system was proved.

Completely similar to the planar case one can relate the oscillator and MICZ-Kepler systems on pseudospheres (two-sheet hyperboloids). In the case of the sphere, the relation between these systems is slightly different. The oscillator on the sphere results in the oscillator on the hyperboloid [16]. After appropriate ‘Wick rotation’ (compare with [17]) of the MICZ-Kepler system on the hyperboloid one can obtain the MICZ-Kepler system on the sphere, constructed in [18].

As far as we know, the integrable (pseudo)spherical analogs of the anisotropic oscillator and the oscillator with the nonlinear potential (3) were unknown up to now, as well as the (pseudo)spherical analog of the (MICZ-)Kepler system with linear and $\cos \theta$ potential terms. The construction of these (pseudo)spherical systems is not only of academic interest, but they could also be useful for the study of the various physical phenomena in nanostructures, as well as in the early universe. For example, the spherical generalization of the anisotropic oscillator potential can be used as the confining potential restricting the motion of particles to the asymmetric segments of the thin (pseudo)spherical films. While with the (pseudo)spherical generalization of the linear potential at hands one can study the impact of the curvature of space on the Stark effect.

The construction of these systems is the goal of the present paper. We shall present the integrable (pseudo)spherical analog of the four-dimensional oscillator with the additional anisotropic potentials (1) and (3), given, respectively, by the expressions

$$\frac{\Delta\omega^2}{2} \mathbf{x}\sigma_3\bar{\mathbf{x}} \tag{5}$$

and

$$\varepsilon_{\text{el}} R_0^2 \frac{(R_0^2 + x_0^2)}{x_0^4} (\mathbf{x}\bar{\mathbf{x}})(\mathbf{x}\sigma_3\bar{\mathbf{x}}). \tag{6}$$

Here $\mathbf{x} = x_\alpha + i x_{\alpha+2}$, $\alpha = 1, 2$, are the Cartesian coordinates of the ambient (pseudo)Euclidean space $\varepsilon \mathbf{x}\bar{\mathbf{x}} + x_0^2 = R_0^2$ and $\sigma_3 = \text{diag}(1, -1)$.

Then, performing the Kustaanheimo–Steffel transformation, we get the integrable Kepler system on the pseudosphere with additional potential terms generalizing linear and $\cos \theta$ potentials of the ordinary (MICZ-)Kepler system.

These potentials can be written as follows:

$$\varepsilon_{\text{el}} \frac{x_0}{R_0} x_3 + \frac{\Delta\omega^2}{2} \left(\frac{x_3}{|\mathbf{x}|} \pm \frac{x_0 x_3}{R_0^2} \right). \tag{7}$$

The upper sign corresponds to the potential reduced from the four-dimensional sphere, and the lower sign corresponds to that reduced from the pseudosphere. We also present the generalization of parabolic coordinates, where the resulting system admits separation of variables. Finally, performing the ‘Wick rotation’ of the latter system we will obtain the spherical analog of the MICZ-Kepler system with linear and $\cos \theta$ potentials: in terms of the ambient space these potentials are defined by the same expressions as the pseudospherical ones (7).

2. Euclidean systems

Let us start from the consideration of the Euclidean analog of our construction. Namely, let us present the integrable four-dimensional anisotropic inharmonic oscillator, and performing the Kustaanheimo–Stiefel transformation, reduce it to the MICZ-Kepler system with linear and $\cos \theta$ potentials. It is convenient to describe the initial four-dimensional system in complex coordinates

$$z^\alpha = \frac{x_1^\alpha + i x_2^\alpha}{\sqrt{2}}, \quad \pi_\alpha = \frac{p_{1|\alpha} - i p_{2|\alpha}}{\sqrt{2}}, \quad \alpha = 1, 2 \tag{8}$$

so that the non-zero Poisson brackets between phase-space coordinates look as follows:

$$\{\pi_\alpha, z^\beta\} = \delta_\alpha^\beta, \quad \{\bar{\pi}_\alpha, \bar{z}^\beta\} = \delta_\alpha^\beta, \quad \alpha, \beta = 1, 2. \tag{9}$$

In these coordinates the Hamiltonian of the isotropic oscillator reads

$$\mathcal{H}_0 = \pi \bar{\pi} + \omega^2 z \bar{z}. \tag{10}$$

Its rotational symmetry generators are defined by the expressions

$$\mathbf{J} = \frac{i}{2} (\pi z - \bar{z} \bar{\pi}), \tag{11}$$

$$\mathbf{J} = \frac{i}{2} (\pi \sigma z - \bar{z} \sigma \bar{\pi}), \tag{12}$$

$$J_{\alpha\beta} = \frac{1}{2} \pi_\alpha \bar{z}^\beta, \quad J_{\bar{\alpha}\bar{\beta}} = \frac{1}{2} \bar{\pi}_\alpha z^\beta, \tag{13}$$

and the hidden symmetry generators read

$$\mathbf{A} = \frac{1}{2} (\pi \sigma \bar{\pi} + \omega^2 \bar{z} \sigma z), \tag{14}$$

$$A_{\alpha\beta} = \frac{1}{2} (\pi_\alpha \pi_\beta + \omega^2 \bar{z}^\alpha \bar{z}^\beta), \quad A_{\bar{\alpha}\bar{\beta}} = \bar{A}_{\beta\alpha}. \tag{15}$$

Here $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli σ -matrices.

Let us note that the whole set of constants of motion: (11)–(15) form the algebra $su(4)$; generators (12), (13) form the algebra $so(4)$ and generator (12) forms the $so(3)$ algebra. For our consideration it is important that generator (11) commutes with (12) and (14) generators only, and these generators form the algebra $so(4)$. Hence, reducing the four-dimensional oscillator by the action of generator (11) we shall get the three-dimensional system with $so(4)$ symmetry algebra.

Let us write the anisotropic inharmonic deformation of the four-dimensional oscillator system given by the Hamiltonian

$$\mathcal{H}_{\text{aosc}} = \mathcal{H}_0 + (\Delta\omega^2 + 2\varepsilon_{el}z\bar{z})z\sigma_3\bar{z}. \tag{16}$$

It has the constants of motion given by (11), the third component of (12) and the hidden symmetry generator

$$A = A_3 + \frac{\Delta\omega^2}{2}(z\bar{z}) + \frac{\varepsilon_{el}}{2}((z\bar{z})^2 + (z\sigma_3\bar{z})^2). \tag{17}$$

It is easy to see that all these constants of motion commute with each other, $\{J, J_3\} = \{J, A\} = \{J_3, J_A\} = \{J, J_3\} = 0$, i.e. Hamiltonian (16) defines the classically integrable system.

Clearly, the potential term (1) decouples the initial isotropic oscillator in the anisotropic one with the frequencies $\omega_{\pm} = \sqrt{\omega^2 \pm \Delta\omega^2}$. The second part of the deformation term given by (3) has no such simple explanation. After the transformation from the initial system to the Kepler-like one it results in the linear potential.

Remark 1. Assuming that z^α are real coordinates we arrive at the two-dimensional anisotropic inharmonic oscillator. More generally, for $\alpha, \beta = 1, \dots, N \geq 2$, and $\hat{\sigma}_3$ is $N \times N$ -dimensional Hermitian matrix which obeys the condition $\hat{\sigma}_3^2 = 1$ we get an integrable anisotropic $4N$ -($2N$ -)dimensional inharmonic oscillator, when z^α are complex (real) coordinates.

Let us perform the Kustaanheimo–Stiefel transformation for the present system. For this purpose we have to reduce the system under consideration by the Hamiltonian action of the $U(1)$ group given by generator (11) and choose the $U(1)$ -invariant coordinates [4, 7]

$$\mathbf{q} = z\sigma\bar{z}, \quad \mathbf{p} = \frac{z\sigma\pi + \bar{\pi}\sigma\bar{z}}{2(z\bar{z})}. \tag{18}$$

As a result, the reduced Poisson brackets read

$$\{p_i, q^j\} = \delta_i^j, \quad \{p_i, p_j\} = s \frac{\varepsilon_{ijk}q^k}{q^3}, \quad q = |\mathbf{q}|, \tag{19}$$

where s is the value of the generator (11), $J = s$. The oscillator’s energy surface, $\mathcal{H}_{\text{aosc}} = E_{\text{aosc}}$, can be presented in the form

$$\mathcal{H}_{\text{MICZ}} = \mathcal{E}_{\text{MICZ}} \tag{20}$$

where

$$\mathcal{H}_{\text{MICZ}} = \frac{\mathbf{p}^2}{2} + \frac{s^2}{2q^2} - \frac{\gamma}{q} + \frac{\Delta\omega^2}{2} \frac{q_3}{q} + \varepsilon_{el}q_3 \tag{21}$$

and

$$\gamma = \frac{E_{\text{aosc}}}{2}, \quad \mathcal{E}_{\text{MICZ}} = -\frac{\omega^2}{2}. \tag{22}$$

It is seen that (19) and (21) define the MICZ-Kepler system with the additional potential (2) in the presence of a constant electric field pointed along the x_3 -axes. For the completeness, let us write the constants of motion of the constructed system by reducing the constants of

motion of the four-dimensional oscillator. The J_3 results in the corresponding component of angular momentum,

$$J = \mathbf{n}_3 \mathbf{J}, \quad \mathbf{J} = \mathbf{p} \times \mathbf{q} + s \frac{\mathbf{q}}{q}. \quad (23)$$

The reduced generator A looks as follows:

$$A = \mathbf{n}_3 \mathbf{A} + \frac{\varepsilon_{\text{el}}}{2} (\mathbf{n}_3 \times \mathbf{q})^2 + \Delta \omega^2 \frac{(\mathbf{n}_3 \times \mathbf{q})^2}{q}, \quad (24)$$

where

$$\mathbf{A} = \mathbf{p} \times \mathbf{J} + \gamma \frac{\mathbf{q}}{q} \quad (25)$$

is the Runge–Lenz vector of the unperturbed MICZ–Kepler system.

Now, we are ready to consider similar oscillator-like systems on the four-dimensional sphere and the pseudosphere, as well as the Kepler-like systems on the three-dimensional pseudosphere.

3. Anisotropic inharmonic Higgs oscillator

For the description of the four-dimensional Higgs oscillator it is convenient to introduce the (complex) projective coordinates connected with the Cartesian coordinates of the five-dimensional ambient space as follows:

$$\mathbf{x}^\alpha \equiv x^\alpha + \iota x^{\alpha+2} = R_0 \frac{2z^\alpha}{1 + \varepsilon z \bar{z}}, \quad x_0 = R_0 \frac{1 - \varepsilon z \bar{z}}{1 + \varepsilon z \bar{z}}, \quad \alpha = 1, 2. \quad (26)$$

Here $\mathbf{x}\bar{\mathbf{x}} + \varepsilon x_0^2 = R_0^2$, with $\varepsilon = 1$ for the sphere and $\varepsilon = -1$ for the two-sheet hyperboloid. In these coordinates the metric of (pseudo)sphere reads

$$ds^2 = \frac{4R_0^2 dz d\bar{z}}{(1 + \varepsilon z \bar{z})^2}, \quad (27)$$

where $|z| \in [0, \infty)$ for the sphere and $|z| \in [0, 1)$ for the pseudosphere. In the limit $R_0 \rightarrow \infty$ the lower hemisphere (the lower sheet of the hyperboloid) converts into a whole two-dimensional plane.

Now, defining the canonical Poisson brackets (9), we can represent the Hamiltonian of the four-dimensional Higgs oscillator as follows:

$$\mathcal{H}_0^\varepsilon = \frac{(1 + \varepsilon z \bar{z})^2 \pi \bar{\pi}}{2R_0^2} + \frac{2\omega^2 R_0^2 z \bar{z}}{(1 - \varepsilon z \bar{z})^2}. \quad (28)$$

The symmetries of (pseudo)sphere are defined by generators (11)–(13), and

$$J_\alpha = (1 - \varepsilon z \bar{z}) \pi_\alpha + \varepsilon (\pi z + \bar{\pi} \bar{z}) \bar{z}^\alpha, \quad J_{\bar{\alpha}} = \bar{J}_\alpha. \quad (29)$$

It is clear that generators (11)–(13) define the rotational symmetry algebra of the Higgs oscillator, while generators (29) define the translations on the (pseudo)sphere. By their use one can construct the generators of hidden symmetries of the Higgs oscillator,

$$A_{\alpha\beta} = \frac{J_\alpha J_\beta}{2R_0^2} + 2\omega^2 R_0^2 \frac{\bar{z}^\alpha \bar{z}^\beta}{(1 - \varepsilon z \bar{z})^2}, \quad A_{\bar{\alpha}\bar{\beta}} = \bar{A}_{\alpha\beta} \quad (30)$$

and

$$\mathbf{A} = \frac{(J \boldsymbol{\sigma} \bar{J})}{2R_0^2} + 2\omega^2 R_0^2 \frac{(z \boldsymbol{\sigma} \bar{z})}{(1 - \varepsilon z \bar{z})^2}. \quad (31)$$

Let us construct the integrable (pseudo)spherical analog of the anisotropic inharmonic oscillator (16). We consider the class of Hamiltonians

$$\mathcal{H}_{\text{aosc}}^\epsilon = \mathcal{H}_0^\epsilon + (z\sigma_3\bar{z})\Lambda(z\bar{z}), \tag{32}$$

which besides the symmetries defined by the generators J and J_3 ($\{J, \mathcal{H}_{\text{aosc}}^\epsilon\} = \{J_3, \mathcal{H}_{\text{aosc}}^\epsilon\} = 0$), possesses the hidden symmetry defined by the constant of motion

$$A = A_3 + g(z\bar{z}) + (z\widehat{\sigma}_3\bar{z})^2 h(z\bar{z}). \tag{33}$$

Here $\Lambda(z\bar{z})$, $g(z\bar{z})$ and $h(z\bar{z})$ are some unknown functions, and A_3 is the third component of (31). It is clear that $\{J, A\} = \{J_3, A\} = 0$ for any choice of, Λ, g, h functions. Hence, requiring A to be the constant of motion, we shall get the integrable anisotropic generalization of the Higgs oscillator.

Surprisingly, from this requirement (that A is the constant of motion, $\{A, \mathcal{H}_{\text{aosc}}^\epsilon\} = 0$) we uniquely (up to constant parameters) define the functions Λ, g, h . Namely, the function Λ in (32) reads

$$\Lambda \equiv \frac{2R_0^2\Delta\omega^2}{(1+\epsilon z\bar{z})^2} + \frac{8\epsilon_{\text{el}}R_0^4}{(1-(z\bar{z})^2)^2} \frac{(1+(z\bar{z})^2)(z\bar{z})}{(1-\epsilon z\bar{z})^2} \tag{34}$$

and the hidden symmetry generator looks as follows,

$$A = A_3 + \frac{2R_0^2\Delta\omega^2 z\bar{z}}{(1+\epsilon z\bar{z})^2} + 4\epsilon_{\text{el}}R_0^4 \left(\frac{(z\bar{z})^2}{(1-(z\bar{z})^2)^2} + \frac{(z\widehat{\sigma}_3\bar{z})^2}{(1-\epsilon z\bar{z})^4} \right). \tag{35}$$

One can easily see that the constructed system results in (16) results in the limit $R_0 \rightarrow \infty$.

Hence, we have got the well-defined (pseudo)spherical generalization of (16).

In coordinates (26) the potential of the constructed system looks much simpler. The potential of (isotropic) Higgs reads

$$U_{\text{Higgs}} = \frac{\omega^2 R_0^2}{2} \frac{R_0^2 - x_0^2}{x_0^2}, \tag{36}$$

while the anisotropy terms are defined by the expression

$$U_{\text{AI}} = \left(\frac{\Delta\omega^2}{2} + \epsilon\epsilon_{\text{el}}R_0^2 \frac{(R_0^4 - x_0^4)}{x_0^4} \right) \mathbf{x}\widehat{\sigma}_3\bar{\mathbf{x}}. \tag{37}$$

4. MICZ-Kepler-like systems on pseudosphere

In this section performing Kustaanheimo–Stiefel transformation on the constructed system we shall get the pseudospherical analog of the Hamiltonian (21).¹ This procedure is completely similar to those of the isotropic Higgs oscillator [16].

First, we must reduce the system by the Hamiltonian action of the generator (11). Choosing the functions (18) as the reduced coordinates and fixing the level surface $J = s$, we shall get the six-dimensional phase space equipped by the Poisson brackets (19). Then we fix the energy surface of the oscillator on the (pseudo)sphere, $\mathcal{H}_{\text{aosc}}^\epsilon = E_{\text{aosc}}$, and multiply it by $(1 - \epsilon q^2)^2/q^2$. As a result, the energy surface of the reduced system takes the form

$$\mathcal{H}_{\text{AMICZ}}^- = \mathcal{E}_{\text{AMICZ}}^-, \tag{38}$$

¹ Let us recall that both *spherical* and *pseudospherical* Higgs oscillators result, upon Kustaanheimo–Stiefel transformation, in the *pseudospherical* MICZ-Kepler system [16].

where

$$\mathcal{H}_{\text{AMICZ}}^- = \frac{(1-q^2)^2}{8r_0^2} \left(\mathbf{p}^2 + \frac{s^2}{q^2} \right) - \frac{\gamma}{2r_0} \frac{1+q^2}{q} + \frac{\Delta\omega^2}{2} \left(\frac{1-\epsilon q}{1+\epsilon q} \right)^2 \frac{q_3}{q} + 2\epsilon_{\text{el}} r_0 \frac{1+q^2}{1-q^2} \frac{q_3}{1-q^2}, \quad (39)$$

$$r_0 = R_0^2, \quad \gamma = \frac{E_{\text{aosc}}}{2}, \quad \mathcal{E}_{\text{AMICZ}}^- = -\frac{\omega^2}{2} + \epsilon \frac{E_{\text{aosc}}}{2r_0}. \quad (40)$$

Interpreting \mathbf{q} as the stereographic coordinates of the three-dimensional pseudosphere

$$\mathbf{x} = r_0 \frac{2\mathbf{q}}{1-q^2}, \quad x_0 = r_0 \frac{1+q^2}{1-q^2}, \quad (41)$$

we conclude that (39) defines the pseudospherical analog of the MICZ-Kepler system with linear and $\cos \theta$ potential terms (21).

The constants of motion of the anisotropic oscillators, J_3 and A yield, respectively, the third component of angular momentum (23) and the hidden symmetry generator

$$A = \mathbf{n}_3 \mathbf{A} + \frac{r_0 \Delta\omega^2}{(1+\epsilon q)^2} \left[\frac{q^2 - q_3^2}{q} \right] + 2\epsilon_{\text{el}} r_0^2 \frac{q^2 - q_3^2}{(1-q^2)^2} \quad (42)$$

where

$$\mathbf{A} = \frac{\mathbf{T} \times \mathbf{J}}{2r_0} + \gamma \frac{\mathbf{q}}{q}$$

is the Runge–Lenz vector of the MICZ-Kepler system on the pseudosphere, \mathbf{J} is the generator of the rotational momentum defined by expression (23) and

$$\mathbf{T} = (1+q^2)\mathbf{p} - 2(\mathbf{q}\mathbf{p})\mathbf{q} \quad (43)$$

is the translation generator.

This term also looks simple in Euclidean coordinates of ambient space

$$V_{\text{AI}} = \frac{\Delta\omega^2}{2} \left(\frac{x_3}{|\mathbf{x}|} + \epsilon \frac{x_0 x_3}{r_0^2} \right) + \epsilon_{\text{el}} \frac{x_0 x_3}{r_0}. \quad (44)$$

Let us note that the term proportional to $\Delta\omega^2$ depends on ϵ , i.e., formally, the anisotropic terms yield different pseudospherical generalizations of potential (2). However, this difference is rather trivial, it is easy to observe that one potential transforms into the other one upon spatial reflection.

The present Kepler-like system admits the separation of variables into the following generalization of parabolic coordinates (compare with [19]):

$$q_1 + iq_2 = \frac{2\sqrt{\xi\eta}}{r_0 + \frac{\sqrt{\sqrt{(r_0^2+\xi^2)(r_0^2+\eta^2)}+\xi\eta+r_0^2}}{\sqrt{2}}} e^{i\varphi}, \quad (45)$$

$$q_3 = \frac{\sqrt{2}\sqrt{\sqrt{(r_0^2+\xi^2)(r_0^2+\eta^2)}-\xi\eta-r_0^2}}{r_0 + \frac{\sqrt{\sqrt{(r_0^2+\xi^2)(r_0^2+\eta^2)}+\xi\eta+r_0^2}}{\sqrt{2}}}.$$

In these coordinates the metric reads

$$ds^2 = r_0^2 \frac{\xi + \eta}{4} \left(\frac{d\xi^2}{\xi(r_0^2 + \xi^2)} + \frac{d\eta^2}{\eta(r_0^2 + \eta^2)} \right) + \xi\eta d\varphi^2. \quad (46)$$

Passing to the canonical momenta, one can represent the Hamiltonian (39) as follows:

$$\begin{aligned} \mathcal{H}_{\text{MICZ}}^- = & \frac{2\xi(r_0^2 + \xi^2)}{r_0^2(\xi + \eta)} p_\xi^2 + \frac{2\eta(r_0^2 + \eta^2)}{r_0^2(\xi + \eta)} p_\eta^2 + \frac{1}{\xi\eta} \frac{p_\varphi^2}{2} \\ & + \frac{sp_\varphi + s^2}{r_0(\xi + \eta)} \left(\frac{r_0 + \sqrt{r_0^2 + \xi^2}}{\xi} + \frac{r_0 - \sqrt{r_0^2 + \eta^2}}{\eta} \right) \\ & + \frac{\Delta\omega^2 r_0}{2} \frac{\xi\sqrt{r_0^2 + \xi^2} - \eta\sqrt{r_0^2 + \eta^2} + \xi^2 - \eta^2}{\xi + \eta} \\ & - \frac{\gamma}{r_0} \frac{\sqrt{r_0^2 + \xi^2} + \sqrt{r_0^2 + \eta^2}}{\xi + \eta} + \varepsilon_{el} \frac{\xi - \eta}{2}. \end{aligned} \tag{47}$$

So, the corresponding generating function has to have the additive form $S = \mathcal{E}_{\text{AMICZ}}t + p_\varphi\varphi + S_1(\xi) + S_2(\eta)$. Replacing p_ξ and p_η by $dS_1(\xi)/d\xi$ and $dS_2(\eta)/d\eta$ respectively, we obtain the following ordinary differential equations:

$$\begin{aligned} \frac{2\xi(r_0^2 + \xi^2)}{r_0^2} \left(\frac{dS_1(\xi)}{d\xi} \right)^2 + (sp_\varphi + s^2) \frac{r_0 + \sqrt{r_0^2 + \xi^2}}{r_0\xi} \\ + \frac{\Delta\omega^2 r_0}{2} (\xi\sqrt{r_0^2 + \xi^2} + \xi^2) - \frac{\gamma}{r_0} \sqrt{r_0^2 + \xi^2} + \varepsilon_{el}\xi^2 - \mathcal{E}_{\text{AMICZ}}\xi + \frac{p_\varphi^2}{\xi} = \beta \end{aligned} \tag{48}$$

$$\begin{aligned} \frac{2\eta(r_0^2 + \eta^2)}{r_0^2} \left(\frac{dS_2(\eta)}{d\eta} \right)^2 + (sp_\varphi + s^2) \frac{r_0 - \sqrt{r_0^2 + \eta^2}}{r_0\eta} - \frac{\Delta\omega^2 r_0}{2} (\eta\sqrt{r_0^2 + \eta^2} + \eta^2) \\ - \frac{\gamma}{r_0} \sqrt{r_0^2 + \eta^2} - \varepsilon_{el}\eta^2 - \mathcal{E}_{\text{AMICZ}}\eta + \frac{p_\varphi^2}{\eta} = -\beta \end{aligned} \tag{49}$$

From these equations we can immediately find the explicit expression for the generating function. We have separated the variables for the pseudospherical generalization of the Coulomb system into linear and $\cos\theta$ potentials.

The above equations look much simpler in the new coordinates (χ, ζ) , where $\xi = r_0 \sinh \chi$, $\eta = r_0 \sinh \zeta$:

$$\begin{aligned} \left(\frac{dS_1(\chi)}{d\chi} \right)^2 = & \frac{\mathcal{E}_{\text{AMICZ}}}{2} - \frac{\Delta\omega^2 r_0^4}{2} (\cosh \chi + \sinh \chi) + \left(\frac{\gamma r_0}{2} - s^2 - sp_\varphi \right) \coth \chi \\ & - \frac{\varepsilon_{el} r_0^3}{2} \sinh \chi - \frac{p_\varphi^2}{2 \sinh^2 \chi} + \frac{\beta r_0 - s^2 - sp_\varphi}{2 \sinh \chi}, \end{aligned} \tag{50}$$

$$\begin{aligned} \left(\frac{dS_2(\zeta)}{d\zeta} \right)^2 = & \frac{\mathcal{E}_{\text{AMICZ}}}{2} + \frac{\Delta\omega^2 r_0^4}{2} (\cosh \zeta + \sinh \zeta) + \left(\frac{\gamma r_0}{2} + s^2 + sp_\varphi \right) \coth \zeta \\ & + \frac{\varepsilon_{el} r_0^3}{2} \sinh \zeta - \frac{p_\varphi^2}{2 \sinh^2 \zeta} - \frac{\beta r_0 + s^2 + sp_\varphi}{2 \sinh \zeta}. \end{aligned} \tag{51}$$

Remark 2. In the same manner the $2p$ -dimensional anisotropic inharmonic oscillator on the (pseudo)sphere can be connected to the $(p + 1)$ -dimensional Kepler-like systems on the pseudosphere also for $p = 1, 4$. For $p = 1$ we should just assume that z^α are real coordinates. In this case we should not perform any reduction at the classical level (in the quantum case we have to reduce the initial system by the discrete Z_2 group action, see [6]). For $p = 4$ we have

to assume that z^α are *quaternionic* coordinates (equivalently that z^α are complex coordinates with $\alpha = 1, \dots, 4$). In contrast to $p = 1, 2$ cases, we should reduce the initial system by the $SU(2)$ group action [8].

Remark 3. The planar (MICZ)-Kepler system with a linear potential can be obtained as a limiting case of the two-center (MICZ-) Kepler system, when one of the forced centers is placed at infinity (see, e.g., [1]). The two-center (pseudo)spherical Kepler system is the integrable system as well [15]. However, the presented pseudospherical generalization of the (MICZ-)Kepler system with linear potential could not be obtained from the two-center pseudospherical Kepler system: it can easily be checked that in contrast to the pseudospherical Kepler potential, it does not obey the corresponding Laplas equation.

5. Transition to the sphere

To get the spherical counterpart of the Hamiltonian (21), let us perform its ‘Wick rotation’

$$\mathbf{q} \rightarrow \iota \mathbf{q}, \quad \mathbf{p} \rightarrow -\iota \mathbf{p}, \quad r_0 \rightarrow -\iota r_0.$$

This transformation yields the following system:

$$\mathcal{H}^+ = \mathcal{H}_0^+ + 2\varepsilon_{\text{el}} r_0 \frac{1-q^2}{1+q^2} \frac{q_3}{1+q^2} + \frac{\Delta\omega^2}{2} \left(\frac{1-i\epsilon q}{1+i\epsilon q} \right)^2 \frac{q_3}{q}, \quad (52)$$

where

$$\mathcal{H}_0^+ = \frac{(1+q^2)^2}{8r_0^2} \left(\mathbf{p}^2 + \frac{s^2}{q^2} \right)^2 - \gamma \frac{1-q^2}{2r_0 q} \quad (53)$$

is the Hamiltonian of the unperturbed MICZ-Kepler system on the sphere. The hidden symmetry of this system is defined by the expression

$$A = \mathbf{n}_3 \mathbf{A} + \Delta\omega^2 \left[\frac{q^2 - q_3^2}{(1+i\epsilon q)^2 q} \right] + 2\varepsilon_{\text{el}} r_0^2 \frac{q^2 - q_3^2}{(1+q^2)^2}, \quad (54)$$

where

$$\mathbf{A} = \mathbf{J} \times \mathbf{T} + \gamma \frac{\mathbf{q}}{q} \quad (55)$$

is the Runge–Lenz vector of the spherical MICZ-Kepler system, with the angular momentum \mathbf{J} given by (23) and with the translation generator

$$\mathbf{T} = (1 - q^2)\mathbf{p} + 2(\mathbf{q}\mathbf{p})\mathbf{q}. \quad (56)$$

One can see that due to the last term in (52) this Hamiltonian is a complex one. Taking its real part we shall get the integrable spherical analog of the MICZ-Kepler system with linear and $\cos \theta$ potentials,

$$\mathcal{H}_{\text{MICZ}}^+ = \mathcal{H}_0^+ + \frac{\Delta\omega^2}{2} \frac{1-6q^2+q^4}{1+q^2} \frac{q_3}{q} + 2\varepsilon_{\text{el}} \frac{1-q^2}{1+q^2} \frac{q_3}{1+q^2}. \quad (57)$$

The generator of its hidden symmetry is also given by the real part of (54)

$$A = \mathbf{n}_3 \mathbf{A} + \left[\Delta\omega^2 r_0 \frac{1-q^2}{q} + \frac{\varepsilon_{\text{el}}}{2} \right] \frac{q^2 - q_3^2}{(1+q^2)^2}. \quad (58)$$

In terms of ambient space \mathbb{R}^4 the anisotropy term is defined by expression (37).

Remark 4. It is clear from our consideration that the addition to the constructed system of the potential

$$c_0 \operatorname{Im} \left(\frac{1 - i\epsilon q}{1 + i\epsilon q} \right)^2 \frac{q_3}{q} \quad (59)$$

will also preserve the integrability. The hidden symmetry generator will be given by the expression

$$A + c_0 \operatorname{Im} \frac{\Delta\omega^2}{2(1 + i\epsilon q)^2} \left[\frac{q^2 - q_3^2}{q} \right]. \quad (60)$$

However, it is easy to see that this additional potential coincides with (52), i.e. we do not get anything new in this way.

6. Summary and conclusion

Let us briefly summarize our results.

- We presented the *integrable (pseudo)spherical generalization of anisotropic oscillator* which can be considered as a deformation of the well-known Higgs oscillator. This integrable deformation is given by potentials (5) and (6). The first potential can be viewed as the (pseudo)spherical analog of the anisotropic oscillator potential (1), and the second one is the (pseudospherical) analog of the inharmonic potential (3).
- Performing the Kustaanheimo transformation of these systems, we constructed the integrable spherical analog of the (three-dimensional) MICZ-Kepler system with linear and $\cos\theta$ potentials. We found the spherical analog of the latter system as well. The (pseudo)spherical analog of the linear (Stark) potential is given by the first term in (7), and that of the $\cos\theta$ potential (2) is given by the second term in (7).

We proved the integrability of these systems postponing the study of its classical and quantum-mechanical solutions. The computation of the quantum-mechanical spectrum of these systems, and, consequently, the clarification of the impact of the space curvature on the Stark effect is a problem of special interest especially an interesting problem from the viewpoint of the mesoscopic physics and the cosmology, as well. Let us note that even in the flat space the presence of the Dirac monopole leads to qualitative changes of the properties of the Stark effect [20]. There is no doubt that similar phenomena will appear in the Stark effect on the curved space. Taking into account the conclusions of recent papers [10, 11], we expect that one can preserve the integrability of the proposed system, introducing the constant magnetic field and the appropriate potential term. In this case we shall have at hands the integrable system in the parallel ‘homogeneous’ electric and magnetic fields. The importance of such a system is obvious.

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